# Technical Comments

## Comments on "Covariance Matrix Approximation"

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THE subject note[1] by Schlegel presented a most useful approximation for error analyses involving trivariate normal distributions. He showed that a conservative estimate for the error volume in the three-dimensional case is always obtained simply by neglecting the covariance elements. Frequently, more than three random errors are to be considered simultaneously. An example would be the three position and velocity deviations from a reference trajectory. The purpose of this note is to prove, in general, that neglect of the covariance elements always yields a conservative error volume estimate and, furthermore, to show that, as the number of error coordinates increases, the approximation becomes more and more conservative.

To begin, the $n$-dimensional Gaussian probability density function for the $n$ performance errors $x_1, x_2, \ldots$, and $x_n$ is written

$$p(\mathbf{x}_n) = \frac{1}{(2\pi)^{n/2}|\mathbf{C}_{nn}|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}_n{}^T\mathbf{C}_{nn}^{-1}\mathbf{x}_n\right) \quad (1)$$

where $\mathbf{C}_{nn}$ is the covariance matrix of the distribution defined as

$$\mathbf{C}_{nn} = \begin{bmatrix} \sigma_1{}^2 & \sigma_{12} & \sigma_{13} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_2{}^2 & \sigma_{23} & \ldots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3{}^2 & \ldots & \sigma_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \ldots & \sigma_n{}^2 \end{bmatrix} \quad (2)$$

Now the determinant of $\mathbf{C}_{nn}$ is a frequently used figure of merit called the generalized variance. The volumes of the equiprobability ellipsoids (ellipsoids with surface contours of constant probability) are directly proportional to the square root of this generalized variance. Incorporating the matrix of correlation coefficients $\rho_{ij} = \sigma_{ij}/\sigma_i\sigma_j$

$$\mathbf{C}_{nn} = \begin{bmatrix} \sigma_1 & & & & & 0 \\ & \sigma_1 & & & & \\ & & \sigma_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \sigma_n \end{bmatrix} \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \ldots & \rho_{1n} \\ \rho_{21} & 1 & \rho_{23} & \ldots & \rho_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_{n1} & \rho_{n2} & & \ldots & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & & 0 \\ & \sigma_2 & & & & \\ & & \sigma_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \sigma_n \end{bmatrix} \quad (3)$$

and

$$|\mathbf{C}_{nn}| = \sigma_1{}^2\sigma_2{}^2\sigma_3{}^2 \ldots \sigma_n{}^2|\boldsymbol{\varrho}_{nn}| = \prod_{j=1}^{n} \sigma_j{}^2|\boldsymbol{\varrho}_{nn}| \quad (4)$$

From physical considerations $\mathbf{C}_{nn}$ is positive definite; hence by (3), $\boldsymbol{\varrho}_{nn}$ is positive definite. By this requirement, the eigenvalues ($\lambda_1, \lambda_2, \ldots$, and $\lambda_n$) of $\boldsymbol{\varrho}_{nn}$ must all be positive.[2]

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Now we wish to prove

$$|\mathbf{C}_{nn}| \leq \prod_{j=1}^{n} \sigma_j{}^2 \quad (5)$$

for in this way the approximate error volume is always larger than the actual volume. Referring to (4), this reduces to the requirement that

$$|\boldsymbol{\varrho}_{nn}| \leq 1 \quad (6)$$

The correlation matrix $\boldsymbol{\varrho}_{nn}$ is an $n \times n$ symmetric matrix. Any matrix of this type can be diagonalized by an orthogonal transformation of the form $\boldsymbol{\Phi}^T \boldsymbol{\varrho}_{nn}\boldsymbol{\Phi}$ where $\boldsymbol{\Phi}$ is the eigenvector matrix formed with the eigenvectors as successive columns. Each eigenvector $\boldsymbol{\phi}_i$ is associated with an eigenvalue $\lambda_i$. Thus

$$\dot{\boldsymbol{\varrho}}_{nn} = \boldsymbol{\Phi}^T\boldsymbol{\varrho}_{nn}\boldsymbol{\Phi} = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & \cdot & \lambda_n \end{bmatrix} \quad (7)$$

Under this type of transformation, the trace and determinant are invariant. Hence

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \sum_{i=1}^{n} \lambda_i = n \quad (8)$$

and

$$|\dot{\boldsymbol{\varrho}}_{nn}| = |\boldsymbol{\varrho}_{nn}| = \lambda_1\lambda_2 \ldots \lambda_n = \prod_{i=1}^{n} \lambda_i \quad (9)$$

Now we seek the eigenvalues such that $|\boldsymbol{\varrho}_{nn}|$ is maximum. Equation (8) is introduced into (9) yielding

$$|\boldsymbol{\varrho}_{nn}| = \prod_{i=1}^{n-1} \lambda_i \left(n - \sum_{i=1}^{n-1} \lambda_i\right) \quad (10)$$

and the partial derivatives are formed:

$$\partial|\boldsymbol{\varrho}_{nn}|/\partial\lambda_j = 0 \qquad j = 1, 2, \ldots, n-1 \quad (11)$$

We obtain, since all the eigenvalues must be positive,

$$n - \sum_{\substack{i=1 \\ (i\neq j)}}^{n-1} \lambda_i - 2\lambda_j = 0 \qquad j = 1, 2, \ldots, n-1 \quad (12)$$

The condition for a maximum is $\partial^2|\boldsymbol{\varrho}_{nn}|/\partial\lambda_j{}^2 < 0$. In this case

$$\frac{\partial^2|\boldsymbol{\varrho}_{nn}|}{\partial\lambda_j{}^2} = -2\prod_{\substack{i=1 \\ (i\neq j)}}^{n-1} \lambda_i \qquad j = 1, 2, \ldots, n-1 \quad (13)$$

and the condition is always satisfied.

Setting $j = 1$, Eq. (12) becomes

$$2\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{n-1} = n \quad (14)$$

and, subtracting the equations for $j = 2, 3, \ldots$, and $n-1$ from (14), we obtain

$$\begin{aligned} \lambda_1 - \lambda_2 &= 0 \\ \lambda_1 - \lambda_3 &= 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ \lambda_1 - \lambda_{n-1} &= 0 \end{aligned} \quad (15)$$

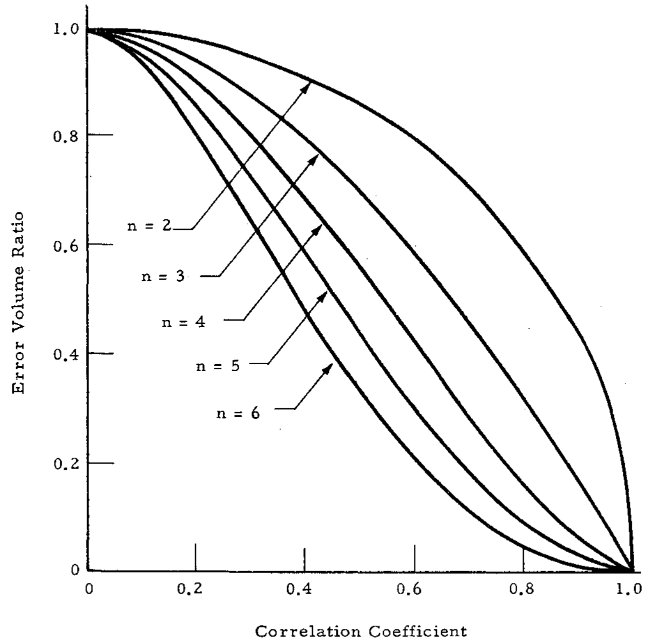


**Fig. 1 Error volume ratio as a function of correlation coefficient.**

Hence $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = 1$ and, using (8), $\lambda_n = 1$. Thus, by (9), the maximum $|\varrho_{nn}|$ is unity; this is obviously the case only when all $\rho_{ij} = 0$. In all other cases when $\rho_{ij} \neq 0$, $|\varrho_{nn}| < 1$. Thus (6) has been proved.

In the paper by Schlegel an error volume ratio was defined as

$$\left(|\mathrm{C}_{nn}| \Big/ \prod_{j=1}^{n} \sigma_j^2\right)^{1/2}$$

By (4), this is equivalent to $|\varrho_{nn}|^{1/2}$ and is simply the ratio of error volumes calculated with and without the covariance elements included. Figure 1 is a plot of this ratio vs the correlation coefficient where, for simplicity, all correlation coefficients are assumed equal. Dimensions from 2 to 6 are shown, and it is observed that as $n$ increases the approximation indeed becomes more conservative.

### References


[1] Schlegel, L. B., "Covariance matrix approximation," AIAA J. **11**, 2672–2673 (1963).

[2] Hildebrand, F. B., *Methods of Applied Mathematics* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1952), p. 46.


# Evaluation of a Temperature Difference Method of Computing Convective Heat Transfer


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### Nomenclature

$c_p$ = specific heat, Btu/lb$_m$°R
$g$ = acceleration due to gravity = 32.2 ft/sec² (lb$_m$/lb$_f$)
$h$ = convective heat-transfer coefficient, Btu/ft² sec°R
$i$ = enthalpy, Btu/lb$_m$
$J$ = Joule's constant = 778 ft lb$_f$/Btu
$k$ = thermal conductivity, Btu/fps °R
$q$ = convective heat transfer, Btu/ft² sec
$r$ = recovery factor
$T$ = temperature, °R
$T_m$ = mean temperature = $[(T_r + T_w)/2]$, – °R
$V$ = velocity, fps
$x$ = reference length, ft
$Re$ = Reynolds number = $\rho Vx/\mu$
$Pr$ = Prandtl number = $\mu c_p/k$

#### *Subscripts*

$T$ = values used in temperature difference method
$i$ = values used in enthalpy difference method
$r$ = recovery
$w$ = wall condition
1 = local condition
$()^*$ = reference condition


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### Discussion

MUCH work has been performed to demonstrate the accuracy of the reference enthalpy method using an enthalpy difference to compute convective heat transfer. However, the writer is not aware of any work that presents a comparison of results obtained using this method to those using a temperature difference method. Even though it is somewhat academic to make this comparison, in some cases temperature differences are used to make approximations of the convective heating, and an estimate of the error involved in such approximation would be valuable. Therefore, an attempt was made to compare the results obtained by the two methods.

The study presented below was performed using flat plate laminar boundary-layer equations because they are most familiar and easiest to work with. A similar procedure could be used to define errors associated with the turbulent boundary-layer equations. The two methods of computing the convective heat transfer are presented, followed by a combination of equations from the two methods to arrive at one expression for comparing results.

### Temperature Difference Method

The temperature difference method of computing convective heat transfer to a surface is based on the equation

$$q_T = h_T(T_r - T_w) \quad (1)$$

In this equation, the recovery temperature $T_r$ is determined from a recovery enthalpy and the local static pressure, using the data of Hanson.[1] The recovery enthalpy is computed from the equation

$$i_r = i_1 + r(V_1^2/2gJ) \quad (2)$$

where the recovery factor $r$ is determined from

$$r = (Pr^*)^{1/2}$$

The convective heat-transfer coefficient $h_T$ of Eq. (1) is computed from the expression

$$h_T = 0.332\ (Re^*)^{0.5}(Pr^*)^{0.333}(k^*/x) \quad (3)$$

which is for laminar flow over a flat plate and is derived in Ref. 2. Here the values followed by the superscript* are evaluated at a reference temperature that is determined from a reference enthalpy defined as

$$i^* = 0.5i_w + 0.28i_1 + 0.22i_r \quad (4)$$

Using the values described previously, we can compute the convective heat transfer by Eq. (1).

### Enthalpy Difference Method

In a similar manner to that just presented we can describe the enthalpy difference method of computing convective heat transfer. It is based on the equation

$$q_i = h_i(i_r - i_w) \quad (5)$$